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partially supported by the ANR project "Théorie Ergodique en Mesure Infinie"

29th June 2007

Abstract. *We are interested in the planar Lorentz gas with finite horizon. Its recurrence comes from a criteria of Conze in [6] and of Schmidt in [14]. Another way to prove recurrence is given by Szász and Varjú in [18]. Total ergodicity follows from these results (see [16] and [13]). In this paper we answer a question of Szász about the asymptotic behaviour of the number of visited cells when the time goes to infinity. It is not more difficult to study the asymptotic of the number of obstacles hit by the particle when the time goes to infinity. We give an estimate for the mean and a result of almost sure convergence. For the simple random walk in $\mathbb{Z}^2$, this question has been studied by Dvoretzky and Erdös in [8]. We adapt the proof of Dvoretzky and Erdös. The lack of independence is compensated by a strong decorrelation result due to Chernov ([5]) and by some extensions of the local limit theorem proved by Szász and Varjú in [18].*

1 Introduction

Let us introduce our model. Since the early work of Sinai ([17]), the billiard systems have been studied by many authors ([1, 2, 3, 4, 9]). In $\mathbb{R}^2$, we consider a finite number of convex open sets $O_1, ..., O_I$, with boundary C^3-smooth and with non null curvature. We repeat these sets $\mathbf{Z}^2$-periodically by setting $U_{a,\ell} = \ell + O_a$ for all $a \in \{1, ..., I\}$ and all $\ell \in \mathbb{Z}^2$. We suppose that the closures of the $U_{a,\ell}$ are pairwise disjoint. For any $\ell \in \mathbb{Z}^2$, we call ℓ-cell the set $\bigcup_{a=1}^{I} \partial U_{a,\ell}$. Let us consider a point particle moving in the domain $Q := \mathbb{R}^2 \setminus \bigcup_{a=1}^{I} \bigcup_{\ell\in\mathbb{Z}^2} U_{a,\ell}$ with unit speed and with elastic reflections off ∂Q. We will consider the **finite horizon case**, i.e. we suppose that the time to wait before the next collision is uniformly bounded. We are interested in the asymptotic behaviour (when n goes to infinity) of the number N_n of cells visited before the nth reflection off ∂Q. We will also study the number N'_n of obsatcles hit before the nth reflection. We prove that the expected value of N_n for a particle starting from the 0-cell is equivalent (as n goes to infinity) to $c\frac{n}{\log(n)}$ for some constant $c > 0$. Moreover we prove that $\frac{\log(n)}{n} N_n$ converges almost everywhere to c.

We will see a link between N_n and the number $\mathcal{T}_+$ of reflections off ∂Q before coming back to the initial cell.

1.1 Billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ and billiard transformation (M, ν, T) in the plane

We call configuration of a particle at some time its position-speed couple. When a reflection occurs, there is coexistence of two configurations : one corresponding to the incident vector and one corresponding to the reflected vector. To avoid ambiguity, we will only consider reflected vectors. Hence the set of configurations (position-speed couples) will be :

$$\mathcal{M}_1 := \{(q, \vec{v}) \in Q \times \mathbb{R}^2 \ : \ \|\vec{v}\| = 1; \ \ q \in \partial Q \ \Rightarrow \ \langle \vec{n}(q), \vec{v} \rangle \geq 0\},$$

with $\vec{n}(q)$ the unit vector normal to ∂Q at $q \in \partial Q$ oriented to the inside of Q. The billiard flow $(Y_t)_t$ is the flow on $\mathcal{M}_1$ such that $Y_t(q, \vec{v}) = (q_t, \vec{v}_t)$ is the configuration at time t of a particle with configuration $(q, \vec{v})$ at time 0. The billiard flow preserves the Lebesgue measure μ_1 on $\mathcal{M}_1$.

Now we only consider reflection times. Let M be the set of reflected vectors off ∂Q :

$$M := \{(q, \vec{v}) \in \partial Q \times \mathbb{R}^2 \ : \ \|\vec{v}\| = 1 \text{ and } \langle \vec{n}(q), \vec{v} \rangle \geq 0\}.$$

The billiard transformation T maps a configuration at a reflection time $x \in M$ to the configuration $T(x) = x'$ corresponding to the next reflection off ∂Q. This transformation preserves the measure ν given by $d\nu(q, \vec{v}) = \cos(\varphi)\, dr d\varphi$, with the parametrisation (a, r, φ, ℓ) of $(q, \vec{v}) \in M$ if $q - (j, k)$ is the point of O_a with curvilinear absciss r and if φ is the angular measure of $(\vec{n}(q), \vec{v})$ taken in $[-\frac{\pi}{2}; \frac{\pi}{2}]$.

We define the function $\tau : M \to [0; +\infty[$ by : $\tau(q, \vec{v}) := \min\{s > 0 \ : \ q + s\vec{v} \in \partial Q\}$. The quantity $\tau(q, \vec{v})$ corresponds to the time before the next reflection off ∂Q. **Here, we suppose that the billiard system has finite horizon**, i.e. $\sup \tau < +\infty$. We already know that this system is recurrent (see [6], [14] and [18]) and that it is totally ergodic (see [16] and [13]). The billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ can be represented as the special flow over (M, ν, T) with roof function τ. Let us explicit this. Let us define $\tilde{\mathcal{M}}_1 := \{(x, s) \ : \ x \in M; \ 0 \leq s < \tau(x)\}$ endowed with the measure $\tilde{\mu}_1$ given by : $d\tilde{\mu}_1(x, s) = d\nu(x) ds$. Let $(\tilde{Y}_t)_t$ be the flow defined on $\tilde{M}_1$ by $\tilde{Y}_t(x, s) = (x, s + t)$ with the identifications $(x, \tau(x)) \equiv (T(x), 0)$. Let $\Delta : \tilde{\mathcal{M}}_1 \to \mathcal{M}_1$ given by : $\Delta((q, \vec{v}), s) = (q + s\vec{v}, \vec{v})$. This bi-measurable function satisfies : $Y_t = \Delta \circ \tilde{Y}_t \circ \Delta^{-1}$ and $\Delta_*(\tilde{\mu}_1) = \mu_1$.

1.2 Billiard transformation in the torus $(\bar{M}, \bar{\nu}, \bar{T})$

The billiard in the torus is got from the billiard in the plane by quotienting the positions by $\mathbb{Z}^2$. More precisely, let us define $\bar{M} = \{(q, \vec{v}) \in M \ : \ q \in \bigcup_{a=1}^I \partial O_a\}$ and $\bar{T} : \bar{M} \to \bar{M}$ with $\bar{T}(q, \vec{v}) = (q', \vec{v}')$ if there exists $\ell \in \mathbb{Z}^2$ such that $T(q, \vec{v}) = (q' + \ell, \vec{v}')$. Let $\bar{\nu}$ be the probability measure on $\bar{M}$ proportional to the restriction of ν to $\bar{M}$. Hence, for the billiard transformation, we will use the probability measure $\bar{\nu}$. We endow M with a metric such that :

$$d(y, y') = |r - r'| + |\varphi - \varphi'|,$$

if $(a, r, \varphi, (0, 0))$ and $(a, r', \varphi', (0, 0))$ are the parametrizations of y and y' respectively. We endow M with its Borel σ-algebra $\mathcal{F}$. The study of this system is complicated by the discontinuities of the transformation $\bar{T}$. But it is known that $\bar{T}$ is C^1-regular on $\bar{M} \setminus (R_0 \cup \bar{T}^{-1}(R_0))$, where the set $R_0 := \{(q, \vec{v}) \in \bar{M} \ : \ \langle \vec{n}(q), \vec{v} \rangle = 0\}$ corresponds to tangent vectors.

Analogously, for the billiard flow, we consider the probability measure μ_1 conditionned to the fact that we start from a position in $]0; 1[^2$. For our results, it is the same to consider the measure μ_1 conditionned to the fact that the last reflection before time 0 was in the $(0,0)$-cell. This corresponds to the probability measure proportional to $\tau\bar{\nu}$ on $\bar{M}$. Indeed, the number of cells visited before the n^{th} reflection off ∂Q does only depend on the configuration at the time of the last reflection before time 0. Hence, we will consider probability measures $H\bar{\nu}$ on $\bar{M}$ with $H = \mathbf{1}$ or $H = \frac{\tau}{\int_M \tau\, d\bar{\nu}}$.

1.3 Results

For any $x \in M$ and any integer $n \geq 1$, we define the number $N_n(x)$ of cells visited before the n-th reflection starting from x :

$$N_n(x) := \#\left\{\ell \in \mathbb{Z}^2 \ :\ \exists m = 1, ..., n\ :\ T^m(x) \in \left(\bigcup_{a=1}^{I} \partial U_{a,\ell}\right) \times \mathbb{R}^2\right\}.$$

The number $N'_n(x)$ of obstacles visited before the n-th reflection for a particle starting with configuration x is given by :

$$N'_n(x) := \#\left\{(a,\ell) \in \{1,...,I\} \times \mathbb{Z}^2\ :\ \exists m = 1,...,n\ :\ T^m(x) \in \partial U_{a,\ell} \times \mathbb{R}^2\right\}.$$

Our first result deals with the expectation of these quantities starting from the $(0,0)$-cell.

Theorem 1 *For $H = \mathbf{1}$ or for $H = \frac{\tau}{\int_M \tau\, d\bar{\nu}}$, we have :*

$$\mathbb{E}_{H\bar{\nu}}[N_n] \sim_{n\to+\infty} 2\pi\sqrt{det(\Sigma^2)}\frac{n}{\log(n)},$$

and

$$\mathbb{E}_{H\bar{\nu}}[N'_n] \sim_{n\to+\infty} 2I\pi\sqrt{det(\Sigma^2)}\frac{n}{\log(n)},$$

where $\Sigma^2 = \lim_{n\to+\infty} Var_{\bar{\nu}}\left(\frac{S_n}{\sqrt{n}}\right)$, where $S_n(\bar{x})$ corresponds to the number of the cell at the n^{th} collision if we start from $\bar{x} \in \bar{M}$.

This result is a consequence of some estimations already got by Dolgopyat, Szász and Varjú in [7]. We will precise the link between our questions and their estimations. We will also detail its short proof that will be refined to prove our second result.

For any $y \in \mathcal{M}_1$ and any $t > 0$, let us define :

$$\tilde{N}_t(y) := \#\left\{\ell \in \mathbb{Z}^2\ :\ \exists s \in]0;t] :\ Y_t(y) \in \left(\bigcup_{a=1}^{I} \partial U_{a,\ell}\right) \times \mathbb{R}^2\right\}$$

and

$$\tilde{N}'_t(y) := \#\left\{(a,\ell) \in \{1,...,I\} \times \mathbb{Z}^2\ :\ \exists s \in]0;t] :\ Y_t(y) \in \partial U_{a,\ell} \times \mathbb{R}^2\right\}.$$

These quantities correspond to the number of cells and of obstacles visited before time t for a particule with configuration y at time 0.

Theorem 2 (Almost sure convergence) *For ν-almost every $x \in M$, we have :*

$$\lim_{n\to+\infty} \frac{\log(n)}{n} N_n(x) = 2\pi\sqrt{det(\Sigma^2)} \quad and \quad \lim_{n\to+\infty} \frac{\log(n)}{n} N'_n(x) = 2\pi I\sqrt{det(\Sigma^2)}.$$

For μ_1-almost every $x \in \mathcal{M}_1$, we have :

$$\lim_{t\to+\infty} \frac{\log(t)}{t} \tilde{N}_t = \frac{2\pi\sqrt{det(\Sigma^2)}}{\int \tau\, d\bar{\nu}} \quad and \quad \lim_{t\to+\infty} \frac{\log(t)}{t} \tilde{N}'_t = \frac{2\pi I\sqrt{det(\Sigma^2)}}{\int \tau\, d\bar{\nu}}.$$

1.4 Tools

As we said briefly in the abstract, we use the scheme of the proof of Dvoretzky and Erdös [8]. We will compensate the lack of independence by two ingredients : a strong decorrelation result (proposition 3) and an extension of the local limit theorem (4). Let us recall that, for almost every point in $\bar{M}$, there exists two unique maximal C^1-curves $\gamma^s(x)$ and $\gamma^u(x)$ such that there exists $\tilde{C} > 0$ and $\tilde{\theta} \in]0;1[$ such that :

- For all integer $n \geq 0$, $\bar{T}^n$ is C^1-regular on a neighbourhood of $\gamma^s(x)$ and the diameter of $\bar{T}^n(\gamma^s(x))$ is bounded from away by $\tilde{C}\tilde{\theta}^n$.
- For all integer $n \geq 0$, $\bar{T}^{-n}$ is C^1-regular on a neighbourhood of $\gamma^u(x)$ and the diameter of $\bar{T}^{-n}(\gamma^u(x))$ is bounded from away by $\tilde{C}\tilde{\theta}^n$.

The curves $\gamma^s(x)$ are called **stable curves** and the curves $\gamma^u(x)$ are called **unstable curves**. Let us recall that, according to Chernov [5] (see the few explanations given in appendix), we have the following result :

Proposition 3 (Strong decorrelation property) *For any $\eta \in]0;1]$ and any integer $m \geq 0$, there exists $C_{(\eta,m)} > 0$ and $\delta_{(\eta,m)} \in]0;1[$ such that, for any measurable bounded functions $f : \bar{M} \to \mathbb{C}$ and $g : \bar{M} \to \mathbb{C}$, for any integer $n \geq 0$, we have :*

$$\left|\mathbb{E}_{\bar{\nu}}[f.g\circ\bar{T}^n] - \mathbb{E}_{\nu}[f]\mathbb{E}_{\nu}[g]\right| \leq C_{(\eta,m)}\left(\|f\|_\infty\|g\|_\infty + C_f^{(\eta,u,m)}\|g\|_\infty + C_g^{(\eta,s,m)}\|f\|_\infty\right)\delta_{(\eta,m)}{}^n,$$

with

$$C_f^{(\eta,u,m)} := \sup_{\gamma^u} \sup_{x,y\in\bar{T}^{-m}(\gamma^u);\ x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^\eta}$$

and

$$C_g^{(\eta,s,m)} := \sup_{\gamma^s} \sup_{x,y\in\bar{T}^{m}(\gamma^s);\ x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^\eta}.$$

A local limit theorem has been established by Szász and Varjú in [18]. We will need some refinements of this result. More precisely, we will use the following result (we refer to [**?**] for its proof) :

Proposition 4 *Let any real number $p > 1$. There exist $C > 0$, $a_0 > 0$ and $K_0 > 0$ such that, for any positive integer k, if B is any measurable set such that, if $x \in B$ then $\gamma^s(x) \subseteq B$, we have :*

$$\left|\bar{\nu}\left(\{S_k = (0,0)\}\cap\bar{T}^{-k}(B)\right) - \frac{\bar{\nu}(B)}{2\pi\sqrt{det(\Sigma^2)}k}\right| \leq \frac{C\bar{\nu}(B)^{1/p}}{k\sqrt{k}};$$

if, moreover, r is any positive integer and if A is a union of connected component of $\bar{M}\setminus\bigcup_{i=0}^{r}\bar{T}^{-i}(R_0)$, then for any $M\in\mathbb{Z}^2$, we have :

$$\left|\bar{\nu}\left(A\cap\bar{T}^{-(k+r)}(B)\cap\{S_{k+r}-S_r=M\}\right)-\frac{\bar{\nu}(A)\bar{\nu}(B)}{\sqrt{det(\Sigma^2)}2\pi k}e^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}M,M\rangle}\right|\le$$

$$\le K_0\left(\frac{\bar{\nu}(B)+\bar{\nu}(A)\bar{\nu}(B)^{1/p}}{k^{3/2}}\left(\frac{\|M\|}{\sqrt{k}}+\frac{\|M\|^3}{k^{3/2}}\right)\exp^{-\frac{a_0}{2k}\langle M,M\rangle}+\frac{\bar{\nu}(B)^{1/p}}{k^2}\right). \qquad (1)$$

This is true in particular if A is $(S_1,...,S_r)$-measurable.

1.5 First calculations and proof of theorem 1

It is easy to see that the billiard system (M,ν,T) is a cylindrical extension of the billiard system $(\bar{M},\bar{\nu},\bar{T})$ by some function $\varphi:\bar{M}\to\mathbb{Z}^2$. For any $(q,\vec{v})\in\bar{M}$ and any $(j,k)\in\mathbb{Z}^2$, we have $T(q+(j,k),\vec{v})=(q'+(j,k)+\varphi(q,\vec{v}),\vec{v}')$ with $(q',\vec{v}')=\bar{T}(q,\vec{v})$ and $T^n(q+(j,k),\vec{v})=\left(q_n+(j+k)+\sum_{\ell=0}^{n-1}\varphi(\bar{T}^\ell(q,\vec{v})),\vec{v}_n\right)$ with $(q_n,\vec{v}_n)=\bar{T}^n(q,\vec{v})$. Let us consider the asymptotic covariance matrix Σ^2 associated to φ :

$$\Sigma^2:=\lim_{n\to+\infty}Cov\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi\circ\bar{T}^k\right).$$

Let us notice that the matrix Σ^2 is invertible. Otherwise, there exists $\alpha\in\mathbb{R}^2$ such that $\left(\langle\alpha,\sum_{\ell=0}^{n-1}\varphi\circ\bar{T}^\ell\rangle\right)_{n\ge1}$ is bounded in $L^2(\bar{\nu})$ and this contradicts the ergodicity of the billiard system in the plane (M,ν,T). For any $(q,\vec{v})\in\bar{M}$ and any $(j,k)\in\mathbb{Z}^2$, we have

$$N_n(q+(j,k),\vec{v})=\#\left\{\sum_{\ell=0}^{m-1}\varphi(\bar{T}^\ell(q,\vec{v}));\ m=1,...,n\right\}.$$

For any $x\in\bar{M}$, let us define $\mathcal{I}_k(x)$ the number of the obstacle (taken in $\{1,...,I\}$) on which $\bar{T}^n(x)$ is based. We have :

$$N'_n(q+(j,k),\vec{v})=\#\left\{\left(\sum_{\ell=0}^{m-1}\varphi(\bar{T}^\ell(q,\vec{v})),\mathcal{I}_m(q,\vec{v})\right);\ m=1,...,n\right\}.$$

Hence, $N_n(q+\ell,\vec{v})$ and $N'_n(q+\ell,\vec{v})$ do not depend on $\ell\in\mathbb{Z}^2$. For any non negative integer m, let us define :

$$S_n:=\sum_{\ell=0}^{m-1}\varphi\circ\bar{T}^\ell \text{ and } X_m:=\varphi\circ\bar{T}^{m-1}.$$

The random variable S_n corresponds to the index of the cell at the n^{th} reflection off ∂Q if we start from the $(0,0)$-cell. We have :

$$N_n(x):=\#\{m=1,...,n\ :\ \forall k=m+1,...,n,\ S_m(x)\neq S_k(x)\}$$

and

$$N'_n(x):=\#\{m=1,...,n\ :\ \forall k=m+1,...,n,\ (S_m(x),\mathcal{I}_m(x))\neq(S_k(x),\mathcal{I}_k(x))\}.$$

We have :

$$\mathbb{E}_{H\bar\nu}\left[N_n\right] = \sum_{m=1}^{n} (H\bar\nu)\left(\{\forall k = m+1, ..., n,\ S_k - S_m \neq (0,0)\}\right)$$

$$= \sum_{m=1}^{n} (H\bar\nu)\left(\{\forall \ell = 1, ..., n-m,\ S_\ell \circ \bar T^m \neq (0,0)\}\right)$$

$$= \sum_{m=1}^{n} \mathbb{E}_{\bar\nu}\left[H \mathbf{1}_{B_{n-m}} \circ \bar T^m\right] = \sum_{m=1}^{n} \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{n-m}} H \circ \bar T^{-m}\right],$$

with $B_p := \left\{x \in \bar M\ :\ \forall \ell = 1, ..., p,\ S_\ell(x) \neq (0,0)\right\}$. In the same way, we get :

$$\mathbb{E}_{H\bar\nu}\left[N'_n\right] = \sum_{m=1}^{n} \mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B'_{n-m}} H \circ \bar T^{-m}\right],$$

with $B'_p := \left\{x \in \bar M\ :\ \forall \ell = 1, ..., p,\ (S_\ell(x), \mathcal{I}_\ell) \neq ((0,0), \mathcal{I}_0)\right\}$. First, let us notice that $\mathbf{1}_{B_{n-m}}$ is constant on each stable curve of $\bar T$. Hence, according to proposition 3, we have :

$$\left|\mathbb{E}_{\bar\nu}\left[\mathbf{1}_{B_{n-m}} H \circ T^{-m}\right] - \bar\nu\left(B_{n-m}\right)\right| \leq C_{(\frac{1}{2},1)} \delta_{(\frac{1}{2},1)}{}^{m} \left(C_H^{(\frac{1}{2},u,1)} + \|H\|_\infty\right).$$

Hence :

$$\sup_n \left| \sum_{m=1}^{n} (H\bar\nu)(\bar T^{-m}(B_{n-m})) - \sum_{m=1}^{n} \bar\nu(B_{n-m}) \right| < +\infty.$$

Moreover we will prove that $(\log(p)\bar\nu(B_p))_{p\geq 1}$ converges to some non negative constant. Hence $\mathbb{E}_{H\bar\nu}[N_n]$ and $\mathbb{E}_{\bar\nu}[N_n]$ are equivalent when n goes to infinity. The same is true for N'_n. Hence it suffices to prove that we have :

$$\lim_{p\to+\infty} \log(p)\bar\nu(B_p) = 2\pi\sqrt{det(\Sigma^2)} \tag{2}$$

and that :

$$\lim_{p\to+\infty} \log(p)\bar\nu(B_p) = 2\pi I\sqrt{det(\Sigma^2)}. \tag{3}$$

Moreover, let us consider the first return time $\mathcal{T}_+$ in the initial cell :

$$\mathcal{T}_+ := \min\left\{m \geq 1\ :\ \sum_{l=0}^{m-1} \varphi \circ \bar T^l = (0,0)\right\}.$$

We have $\bar\nu(\mathcal{T}_+ > n) = \bar\nu(B_n)$ and :

$$\mathbb{E}_{\bar\nu}[\min(\mathcal{T}_+, (n+1))] = \sum_{m=0}^{n} \bar\nu(\mathcal{T}_+ > m) = 1 + \sum_{m=1}^{n} \bar\nu(B_m) = 1 + \mathbb{E}_{\bar\nu}\left[N_n\right].$$

Hence our calculations here are linked with estimations for the mean return time to the initial cell.

Such estimations have already been done by Dolgopyat, Szász and Varjú in [7]. However, since the proof is short and since we will complicate this calculation to prove out theorem 2, the proof of (3) can be seen as an introduction to the proof of theorem 2. We will use an idea of Dvoretzky and Erdös. First it is easy to see that :

$$1=\sum_{k=1}^{n}\bar{\nu}\left(\{S_k=(0,0) \text{ and } \forall \ell=k+1,...,n,\ S_l-S_k\neq(0,0)\}\right).$$

Hence we have : $1=\sum_{k=1}^{n}\bar{\nu}\left(\{S_k=(0,0)\}\cap\bar{T}^{-k}(B_{n-k})\right)$.

In the same way, we have :

$$\forall a=1,...,I,\quad 1=\sum_{k=1}^{n}\bar{\nu}\left(\{S_k=(0,0)\}\cap\bar{T}^{-k}(\{\mathcal{I}_0=a\}\cap B'_{n-k})\right).$$

Here, the independence property used by Dvoretzky and Erdös will be replaced by our first extension of the local limit theorem proved by Szász and Varjú. According to the first point of our proposition 4, we know that there exists $C>0$ such that for all non negative integers n and k with $2\le k\le n-1$, we have :

$$\left|\bar{\nu}\left(\{S_k=(0,0)\}\cap\bar{T}^{-k}(B_{n-k})\right)-\bar{\nu}(B_{n-k})\frac{1}{2\pi\sqrt{det(\Sigma^2)}k}\right|\le\frac{C}{k\sqrt{k}}.$$

For any $\varepsilon>0$, we consider an integer ℓ_ε such that : $\sum_{k\ge\ell_\varepsilon}\frac{C}{k\sqrt{k}}<\varepsilon$.

- To get an upper bound, we write : $1\ge\sum_{k=\ell_\varepsilon}^{n}\bar{\nu}\left(\{S_k=(0,0)\}\cap\bar{T}^{-k}(B_{n-k})\right)$ and therefore :

  $$1+\varepsilon\ge\sum_{k=\ell_\varepsilon}^{n}\bar{\nu}(B_{n-k})\frac{1}{2\pi\sqrt{det(\Sigma^2)}k}\ge\bar{\nu}(B_n)\frac{1}{2\pi\sqrt{det(\Sigma^2)}}\sum_{k=\ell_\varepsilon}^{n}\frac{1}{k}.$$

  Hence we have :

  $$\limsup_{n\rightarrow+\infty}\log(n)\bar{\nu}(B_n)\le2\pi\sqrt{det(\Sigma^2)}.$$

- To get a lower bound, we write :

  $$1\le\sum_{k=1}^{\ell_\varepsilon-1}\bar{\nu}(B_{n-k})+\varepsilon+\sum_{k=\ell_\varepsilon}^{m-1}\bar{\nu}(B_{n-k})\frac{1}{2\pi\sqrt{det(\Sigma^2)}k}+\sum_{k=m}^{n}\frac{1}{2\pi\sqrt{det(\Sigma^2)}k}.$$

  Hence we have :

  $$1-\varepsilon\le\bar{\nu}\left(B_{n-m}\right)\left(\ell_\varepsilon+\frac{\sum_{k=\ell_\varepsilon}^{m-1}\frac{1}{k}}{2\pi\sqrt{det(\Sigma^2)}}\right)+\frac{\sum_{k=m}^{n}\frac{1}{k}}{2\pi\sqrt{det(\Sigma^2)}}.$$

  Let us consider an integer $q\ge e^2$. Applying the previous inequality with $n:=q\lfloor\log(q)\rfloor$ and with $m:=q\left(\lfloor\log(q)\rfloor-1\right)$, we get :

  $$1-\varepsilon\le\bar{\nu}\left(B_q\right)\left(\ell_\varepsilon+\frac{\log(q)+\log(\log(q))}{2\pi\sqrt{det(\Sigma^2)}}\right)+\frac{-\log\left(1-\frac{1}{\lfloor\log(q)\rfloor}\right)}{2\pi\sqrt{det(\Sigma^2)}}.$$

  We conclude that :

  $$\liminf_{q\rightarrow+\infty}\log(q)\bar{\nu}(B_q)\ge2\pi\sqrt{det(\Sigma^2)}.$$

Hence we get estimate (3). Let us notice that the same argument gives the same estimate for $\{\mathcal{I}_0=a\}\cap B'_p)$ instead of B_p for all $a=1,...,I$.

2 Proof of theorem 2

The second point is a consequence of the first point. Indeed we have $\tilde{N}_t(x) = N_{n(t+s,y)}(y)$ where y is the configuration of the particle at the previous reflection time, i.e. $\Delta(y,s) = x$ where Δ is the map defined in section 1.1 with $n(t,\cdot)$ the number of reflections before time t : $n(t,x) := \max\{m \geq 0 \ : \ \sum_{k=0}^{m-1} \tau \circ T^k(x) \leq t\}$. We know that $\frac{n(t,\cdot)}{t}$ converges μ_1-almost everywhere to $\frac{1}{\int \tau \, d\bar{\nu}}$ as t goes to $+\infty$. This gives the second point.

To prove the first point it is enough to prove that it is true for $\bar{\nu}$-almost every point in $\bar{M}$. This is what we prove here. The sketch of our proof follows sections 3 and 5 of [8]. In this section we do not give all the details of the proof. We insist on the adaptation to do in order to adapt the proof of Dvoretzki and Erdös of [8].

2.1 Speed of convergence of $\frac{\log(n)}{n}\mathbb{E}_{\bar{\nu}}[N_n]$ and of $\frac{\log(n)}{n}\mathbb{E}_{\bar{\nu}}[N'_n]$

To estimate the variances of N_n and of N'_n, we need a more precise estimate of the expectation of N_n. Here again, we adapt the argument of Dvoretzky and Erdös by replacing the independence property by our first extension of Szász and Varjú local limit theorem (first point of proposition 4). Let us take $L_n := \lceil (\log(n))^2 \rceil$. For n large enough, we have : $1 \leq L_n \leq n$ and :

$$1 \geq \sum_{k=L_n}^{n} \bar{\nu}(B_{n-k}) \frac{1}{2\pi\sqrt{det(\Sigma^2)}k} - C\sum_{k=L_n}^{n} \frac{1}{k\sqrt{k}} \geq \bar{\nu}(B_n) \frac{\log(n) - \log(L_n)}{2\pi\sqrt{det(\Sigma^2)}} - C\left(\frac{1}{\sqrt{L_n}} - \frac{1}{\sqrt{n}}\right).$$

Hence we have :

$$\begin{aligned}
\bar{\nu}(B_n) &\leq 2\pi\sqrt{det(\Sigma^2)} \frac{1 + \frac{C}{\sqrt{L_n}}}{\log(n) - \log(L_n)} \\
&\leq 2\pi\sqrt{det(\Sigma^2)} \frac{1}{\log(n) - \log(L_n)} + O\left(\frac{1}{\sqrt{L_n}\log(n)}\right) \\
&\leq 2\pi\sqrt{det(\Sigma^2)} \frac{1}{\log(n)} + O\left(\frac{\log(L_n)}{(\log(n))^2}\right) + O\left(\frac{1}{\sqrt{L_n}\log(n)}\right) \\
&\leq \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{\log(\log(n))}{(\log(n))^2}\right).
\end{aligned}$$

On the other hand, there exists $K_0 > 0$ such that, for $1 \leq L \leq m-2 \leq m \leq n$, we have :

$$\begin{aligned}
1 &= \sum_{k=1}^{n} \bar{\nu}(\{S_k = (0,0)\} \cap \bar{T}^{-k}(B_{n-k})) \\
&\leq \sum_{k=1}^{L} \frac{K_0 \bar{\nu}(B_{n-k})^{\frac{1}{p}}}{k} + \sum_{k \geq L+1} \frac{C}{k\sqrt{k}} + \sum_{k=L+1}^{m-1} \frac{\bar{\nu}(B_{n-k})}{2\pi\sqrt{det(\Sigma^2)}k} + \sum_{k=m}^{n} \frac{1}{2\pi\sqrt{det(\Sigma^2)}k} \\
&\leq \bar{\nu}(B_{n-m}) \left(C'_0 \log(L)\log(n-m)^{1-\frac{1}{p}} + \frac{\log(m-1)}{2\pi\sqrt{det(\Sigma^2)}}\right) + \frac{2C}{\sqrt{L}} + \sum_{k=m}^{n} \frac{1}{2\pi\sqrt{det(\Sigma^2)}k}.
\end{aligned}$$

Hence we have :

$$\bar{\nu}(B_{n-m}) \geq \frac{1 - \frac{2C}{\sqrt{L}} - \frac{\log(n)-\log(m)}{2\pi\sqrt{det(\Sigma^2)}}}{C'_0 \log(L)\log(n-m)^{1-\frac{1}{p}} + \frac{\log(m)}{2\pi\sqrt{det(\Sigma^2)}}}.$$

Let q be an integer large enough. We take : $n := q\lfloor \log(q) \rfloor$ and $m := q(\lfloor \log(q) \rfloor - 1)$ and $L := \left\lfloor \left(\frac{\log(q)}{\log(\log(q))} \right)^2 \right\rfloor$. We have :

$$\begin{aligned}
\bar{\nu}(B_q) &\geq \frac{1 - \frac{2C}{\sqrt{L}} + O\left(\frac{1}{\log(q)}\right)}{C_0' \log(L)\log(q)^{1-\frac{1}{p}} + \frac{\log(q)+\log(\log(q))}{2\pi\sqrt{det(\Sigma^2)}}} \\
&\geq \frac{1}{C_0' \log(L)\log(q)^{1-\frac{1}{p}} + \frac{\log(q)+\log(\log(q))}{2\pi\sqrt{det(\Sigma^2)}}} + O\left(\frac{\log(\log(n))}{(\log(n))^2}\right) \\
&\geq \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(q)} + O\left(\frac{\log(\log(q))}{(\log(q))^{1+\frac{1}{p}}}\right).
\end{aligned}$$

Hence, we have :

$$\bar{\nu}(B_n) = \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{\log(\log(n))}{(\log(n))^{1+\frac{1}{p}}}\right)$$

and therefore :

$$\mathbb{E}_{\bar{\nu}}[N_n] = 2\pi\sqrt{det(\Sigma^2)}\frac{n}{\log(n)} + O\left(\frac{n\log(\log(n))}{(\log(n))^{1+\frac{1}{p}}}\right).$$

These estimations are true for all real number $p > 1$. Analogously we get :

$$\forall i = 1, ..., I, \quad \bar{\nu}(\{\mathcal{I}_0 = i\} \cap B_n') = \frac{2\pi\sqrt{det(\Sigma^2)}}{\log(n)} + O\left(\frac{\log(\log(n))}{(\log(n))^{1+\frac{1}{p}}}\right)$$

and therefore :

$$\mathbb{E}_{\bar{\nu}}[N_n'] = 2I\pi\sqrt{det(\Sigma^2)}\frac{n}{\log(n)} + O\left(\frac{n\log(\log(n))}{(\log(n))^{1+\frac{1}{p}}}\right).$$

2.2 Estimation of the variance of N_n

We follow the proof of Dvoretzky and Erdös. We use the estimation of $\mathbb{E}_{\bar{\nu}}[N_n]$ obtained in the previous subsection and we replace the independence property by the strong decorrelation property coming from Chernov's calculations (proposition 3).

Let us recall that $N_n = \sum_{m=1}^{n} \mathbf{1}_{B_{n-m}} \circ \bar{T}^m$. Let us take $m_n := \lfloor \sqrt{n} \rfloor$. We have :

$\mathbb{E}_{\bar{\nu}}[(N_n)^2] = \sum_{i,j=1}^{n} \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B_{n-i}} \circ \bar{T}^i \mathbf{1}_{B_{n-j}} \circ \bar{T}^j\right]$

$$\begin{aligned}
&\leq 2n(m_n+1) + 2\sum_{1\leq i\leq i+m_n+1\leq j\leq n} \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B_{j-m_n-i}} \circ \bar{T}^i \mathbf{1}_{B_{n-j}} \circ \bar{T}^j\right] \\
&\leq 2n(m_n+1) + 2\sum_{1\leq i\leq i+m_n+1\leq j\leq n} \mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{B_{j-m_n-i}} \circ \bar{T}^{-(j-m_n-i)} \mathbf{1}_{B_{n-j}} \circ \bar{T}^{m_n}\right] \\
&\leq 2n(m_n+1) + 2\sum_{1\leq i\leq i+m_n+1\leq j\leq n} \bar{\nu}\left(B_{j-i-m_n}\right)\bar{\nu}\left(B_{n-j}\right) + 2C_{(1,0)}\sum_{1\leq i\leq i+m_n\leq j\leq n} {\delta_{(1,0)}}^{m_n} \\
&\leq 2n(m_n+1) + 2n^2 C_{(1,0)}{\delta_{(1,0)}}^{m_n} + 2\sum_{i'=1}^{n}\bar{\nu}(B_{i'})\sum_{j'=1}^{n-i'}\bar{\nu}(B_{j'}),
\end{aligned}$$

according to proposition 3, since $\mathbf{1}_{B_{j-m_n-i}}\circ\bar{T}^{-(j-m_n-i)}$ is constant along the unstable curves of $\bar{T}$ and since $\mathbf{1}_{B_{n-j}}$ is constant along the stable curves of $\bar{T}$. Hence we have :

$$\begin{aligned} Var(N_n) &\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\sum_{i=1}^n\bar{\nu}(B_i)\sum_{j=1}^{n-i}\bar{\nu}(B_j)-\sum_{i'=1}^n\sum_{j'=1}^n\bar{\nu}(B_{i'})\bar{\nu}(B_{j'}) \\ &\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+n+2\sum_{i=1}^n\bar{\nu}(B_i)\left(\sum_{j=1}^{n-i}\bar{\nu}(B_j)-\sum_{j'=i}^n\bar{\nu}(B_{j'})\right).\end{aligned}$$

Let us recall that if $j\le j'$ then $\bar{\nu}(B_{j'})\le\bar{\nu}(B_j)$. Hence, if $j>\lfloor n/2\rfloor$, $\bar{\nu}(B_j)-\bar{\nu}(B_{n-j})\le 0$. Therefore, for any $i=1,...,n$, we have :

$$\begin{aligned} \sum_{j=1}^{n-i}\bar{\nu}(B_j)-\sum_{j'=i}^n\bar{\nu}(B_{j'}) &= -\bar{\nu}(B_n)+\sum_{j=1}^{n-i}(\bar{\nu}(B_j)-\bar{\nu}(B_{n-j})) \\ &\le -\bar{\nu}(B_n)+\sum_{j=1}^{\max(n-i,\lfloor n/2\rfloor)}(\bar{\nu}(B_j)-\bar{\nu}(B_{n-j})) \\ &\le -\bar{\nu}(B_n)+\sum_{j=1}^{\lfloor n/2\rfloor}(\bar{\nu}(B_j)-\bar{\nu}(B_{n-j})).\end{aligned}$$

We get :

$$\begin{aligned} Var(N_n) &\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\sum_{i=1}^n\bar{\nu}(B_i)\left(\sum_{j=1}^{\lfloor n/2\rfloor}\bar{\nu}(B_j)-\sum_{j'=n-\lfloor n/2\rfloor}^{n}\bar{\nu}(B_{j'})\right) \\ &\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+2\mathbb{E}_{\bar{\nu}}[N_n]\left(\mathbb{E}_{\bar{\nu}}[N_{\lfloor n/2\rfloor}]+\mathbb{E}_{\bar{\nu}}[N_{n-\lfloor n/2\rfloor}]-\mathbb{E}_{\bar{\nu}}[N_n]\right) \\ &\le 2n(m_n+1)+2n^2C_{(1,0)}{\delta_{(1,0)}}^{m_n}+O\left(\frac{n}{\log(n)}\right)\times O\left(n\frac{\log(\log(n))}{(\log(n))^{1+\frac{1}{p}}}\right) \\ &\le O\left(n^2\frac{\log(\log(n))}{(\log(n))^{2+\frac{1}{p}}}\right).\end{aligned}$$

Inthe same way, we have :

$$\forall p>1,\quad Var(N'_n)=O\left(n^2\frac{\log(\log(n))}{(\log(n))^{2+\frac{1}{p}}}\right).$$

2.3 End of the proof

In the following n will be such that : $n\ge 1$, $\log(n)\ge 1$ and $\log(\log(n))\ge 1$ and $\log(n/\log(n)^2)\ge((\log(n))/2)$. According to the beginning of section 5 of [8], it suffices to prove that :

$$\exists\delta>0,\ \forall\varepsilon>0,\ \bar{\nu}\left(\left\{\left|N_n-\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right|>2\pi\sqrt{det(\Sigma^2)}\varepsilon\frac{n}{\log(n)}\right\}\right)=O\left((\log(n))^{-1-\delta}\right).$$

and that the same holds for

$$N'_n(a):=\#\{\ell\in\mathbb{Z}^2\ :\ \exists m=1,...,n\ :\ T^m(x)\in\partial U_{a,\ell}\times\mathbb{R}^2\}$$

instead of N_n. As in [8], we consider $L=L_n$ with $L=O(\log(n))$.

1. First we estimate : $\bar\nu\left(\left\{N_n > (1+\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right)$. Let us define, for all $i=1,...,L$:

$$A_i := \left\{N_{\lfloor in/L\rfloor} - N_{\lfloor (i-1)n/L\rfloor} > \left(1+\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}.$$

and :

$$\tilde A_i := \left\{N_{\lfloor in/L\rfloor} - N_{\lfloor (i-1)n/L\rfloor} > \frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}.$$

As noticed by Dvoretzky and Erdös, we have : We have :

$$\left\{N_n > (1+\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\} \subseteq \left(\bigcup_{1\le i<j\le L}(A_i\cap A_j)\right)\cup\left(\bigcup_{1\le i\le L}\tilde A_i\right).$$

(a) We estimate $\bar\nu(A_i)$ using the Markov inequality.
Let us notice that, if $a\le b$, $N_b - N_a$ is less than the number of cells visited between the $(a+1)$th reflection and the bth reflection. Hence, we have :

$$\begin{aligned}\bar\nu\left(A_i\right) &\le \bar\nu\left(\left\{N_{\lfloor n/L\rfloor+1} > \left(1+\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}\right)\\ &\le \bar\nu\left(\left\{N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} > \frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}\right)\\ &\le \frac{\mathbb{E}_{\bar\nu}\left[\left(N_{\lfloor n/L\rfloor+1} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)^2\right]}{\left(\frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)^2}\\ &\le O\left(\frac{\log(\log(n))}{(\log(n))^{\frac{1}{p}}}\right).\end{aligned}$$

Hence, we have :

$$\bar\nu\left(\bigcup_{i=1}^{L}A_i\right) \le \sum_{i=1}^{L}\bar\nu(A_i) \le \frac{L\log(\log(n))}{\log(n)^{\frac{1}{p}}}.$$

(b) Let us estimate $\sum_{1\le i<j\le L}\bar\nu(A_i\cap A_j)$. Here A_i and A_j are not independent (as they were in [8]). We will use the strong decorrelation property (proposition 3).
We have :

$$\sum_{1\le i<j\le L}\bar\nu(A_i\cap A_j) \le \sum_{1\le i<j\le L}\mathbb{E}_{\bar\nu}\left[\mathbf{1}_{A_i}\mathbf{1}_{A'_j}\circ\bar T^{\lfloor\sqrt{n}\rfloor}\right]$$

with $A'_j := \left\{N_{\lfloor jn/L\rfloor} - N_{\lfloor (j-1)n/L\rfloor} - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} > \frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} - \sqrt{n}\right\}$

$$\begin{aligned}\sum_{1\le i<j\le L}\bar\nu(A_i\cap A_j) &\le \sum_{1\le i<j\le L}\mathbb{E}_{\bar\nu}\Big[\mathbf{1}_{A_i}\circ\bar T^{-\lfloor in/L\rfloor}\\ &\qquad\left(\mathbf{1}_{A'_j}\circ\bar T^{-\lfloor (j-1)n/L\rfloor}\right)\circ\bar T^{\lfloor (j-1)n/L\rfloor-\lfloor in/L\rfloor+\sqrt{n}}\Big]\\ &\le \left(\sum_{1\le i<j\le L}\bar\nu\left(A_i\right)\bar\nu\left(A'_j\right)\right) + L^2C_{(1,0)}\delta_{(1,0)}{}^{\sqrt{n}-1},\end{aligned}$$

according to proposition 3 since $\mathbf{1}_{A_i}\circ\bar{T}^{-\lfloor in/L\rfloor}$ is constant along the unstable curves and $\mathbf{1}_{A'_j}\circ\bar{T}^{-\lfloor (j-1)n/L\rfloor}$ is constant along the stable curves. Let us notice that, for n large enough, we have :

$$A'_j\subseteq\left\{N_{\lfloor jn/L\rfloor}-N_{(j-1)n/L}-\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}>\frac{\varepsilon}{4}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right\}.$$

Hence, we get an estimation for $\bar\nu(A'_j)$ analogous to the one obtained for $\bar\nu(A_j)$. We get :

$$\sum_{1\le i<j\le L}\bar\nu\left(A_i\cap A_j\right)=O\left(\frac{L^2(\log(\log(n)))^2}{(\log(n))^{2/p}}\right).$$

(c) We estimate $\bar\nu(\tilde A_i)$ as we did for A_i :

$$\begin{aligned}\bar\nu(\tilde A_i)\ &\le\ \bar\nu\left(\left\{N_{\lfloor n/L\rfloor+1}-\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}>\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\left(\frac{\varepsilon L}{2}-1\right)\right\}\right)\\ &\le\ O\left(\frac{\log(\log(n))}{L^2\log(n)^{\frac1p}}\right).\end{aligned}$$

From this, we get : $\bar\nu\left(\bigcup_{i=1}^L\tilde A_i\right)\le\frac{\log(\log(n))}{L\log(n)^{\frac1p}}$.

(d) Hence, with the choice $L=\left\lfloor\frac{(\log(n))^{\frac13}}{(\log(\log(n)))^{\frac13}}\right\rfloor$ (here only), we get :

$$\begin{aligned}\bar\nu\left(\left\{N_n>(1+\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right)\ &\le\ O\left(\frac{L^2(\log(\log(n)))^2}{(\log(n))^{\frac2p}}\right)+O\left(\frac{\log(\log(n))}{L\log(n)^{\frac1p}}\right)\\ &\le\ O\left(\left(\frac{\log(\log(n))}{\log(n)}\right)^{4/3}\right)\end{aligned}$$

2. Second, we estimate : $\bar\nu\left(\left\{N_n<(1-\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right)$. From now, we will take : $L=\lfloor\log(\log(n)))\rfloor$ (for $n\ge3$).
For any $i=1,...,L$, we set M_i the number of cells visited between times $\lfloor(i-1)n/L\rfloor+1$ and $\lfloor in/L\rfloor$. We define $D_i:=\{M_i<\left(1-\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\}$.
For any $1\le i<j\le L$, we consider the number $M_{i,j}$ of points in common in $\{S_{\lfloor n(i-1)/L\rfloor+1},...S_{\lfloor in/L\rfloor}\}$ and in $\{S_{\lfloor n(j-1)/L\rfloor+1},...S_{\lfloor nj/L\rfloor}\}$.
Let us take $\eta\in]0;1[$. Let us define :

$$C_{i,j}:=\left\{M_{i,j}>\frac{n\log(\log(n))}{L(\log(n))^{1+\eta}}\right\}.$$

As noticed by Dvoretzky and Erdös, we have :

$$\left\{N_n<(1-\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\subseteq\left(\bigcup_{i<j}D_i\cap D_j\right)\cup\left(\bigcup_{i,j,i',j'\ :\ \#\{i,j,i',j'\}=4}(C_{i,j}\cap C_{i',j'})\right).$$

(a) We estimate $\bar{\nu}(D_i)$ thanks to the Markov inequality. We have :

$$\begin{aligned}
\bar{\nu}(D_i) &\leq \bar{\nu}\left(M_i - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} < -\frac{\varepsilon}{2}\frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right) \\
&\leq \frac{\mathbb{E}_{\bar{\nu}}\left[\left(M_i - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)}\right)^2\right]}{\frac{\varepsilon^2 4\pi^2 det(\Sigma^2)n^2}{4L^2\log(n)}} \\
&\leq O\left(n^2\frac{\log(\log(n/L))}{L^2(\log(n/L))^{2+\frac{1}{p}}}\right)\frac{4L^2\log(n)}{\varepsilon^2 4\pi^2 det(\Sigma^2)n^2} \\
&\leq O\left(\frac{\log(\log(n/L))}{(\log(n/L))^{1+\frac{1}{p}}}\right).
\end{aligned}$$

We want to estimate $\sum_{1\leq i<j\leq L}\bar{\nu}(D_i\cap D_j)$. The events D_i and D_j are not independent here but we can use the strong decorrelation property stated in our proposition 3. Moreover, we have :

$$\sum_{1\leq i<j\leq L}\bar{\nu}(D_i\cap D_j) \leq \sum_{1\leq i<j\leq L}\bar{\nu}(D_i\cap D'_j\circ\bar{T}^{\lfloor\sqrt{n}\rfloor}),$$

with $D'_j := \left\{M_j - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} < -\frac{\varepsilon}{2}2\pi\sqrt{det(\Sigma^2)}\frac{n}{L\log(n)} + 2\sqrt{n}\right\}$. Let us notice that, for n large enough, we have :

$$D'_j \subseteq \left\{M_j - \frac{2\pi\sqrt{det(\Sigma^2)}n}{L\log(n)} < -\frac{\varepsilon}{4}2\pi\sqrt{det(\Sigma^2)}\frac{n}{L\log(n)}\right\}.$$

Hence we can estimate $\bar{\nu}(D'_j)$ as we have estimated $\bar{\nu}(D_i)$. According to proposition 3, we have :

$$\begin{aligned}
\sum_{1\leq i<j\leq L}\bar{\nu}(D_i\cap D_j) &\leq \sum_{1\leq i<j\leq L}\bar{\nu}(D_i)\bar{\nu}(D'_j) + \sum_{1\leq i<j\leq L}C_{(1,0)}\delta_{(1,0)}{}^{\sqrt{n}-1} \\
&\leq O\left(\frac{L^2\log(\log(n/L))^2}{(\log(n/L))^4}\right).
\end{aligned}$$

(b) We will in some way replace $M_{i,j}$ by $\tilde{M}_{i,j}$ the cardinal of the common values of $\{S_{\lfloor(i-1)n/L\rfloor+\lceil n/(\log(n))^2\rceil},...,S_{\lfloor in/L\rfloor-\lfloor n/(\log(n))^2\rfloor}\}$ and of $\{S_{\lfloor(j-1)n/L\rfloor+\lceil n/(\log(n))^2\rceil},...,S_{\lfloor jn/L\rfloor-\lfloor n/(\log(n))^2\rfloor}\}$. Let us notice that we have : $M_{i,j}\leq\tilde{M}_{i,j}+O\left(n/(\log(n))^2\right)$. We will use the following notations :

$$\xi_{a,b} := \{\forall q=a,...,b,\ S_q\neq S_{b+1}\} \text{ and } \zeta_{a,b} := \{\forall q=a,...,b,\ S_q\neq S_{a-1}\}.$$

To simplify the expressions, we will use the following notations :

$$n_{(i,-)} := \lfloor(i-1)n/L\rfloor + \lceil n/(\log(n))^2\rceil \quad\text{and}\quad n_{(i,+)} := \lfloor in/L\rfloor - \lfloor n/(\log(n))^2\rfloor.$$

(c) Let us prove that we have :

$$\sup_{i<j}\mathbb{E}_{\bar{\nu}}[M_{i,j}] = O\left(\frac{n\log(\log(n))}{L(\log(n))^2}\right) = O\left(\frac{n}{L(\log(n))^2}\right). \tag{4}$$

Let $i<j$. According to formula 1, we have :
$\mathbb{E}_{\bar\nu}[\tilde M_{i,j}] =$

$$\begin{aligned}
&= \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{\ell=n_{(j,-)}}^{n_{(j,+)}} \bar\nu\left(\xi_{\lfloor (i-1)n/L\rfloor+1,k-1}\cap\{S_k=S_\ell\}\cap\zeta_{\ell+1,\dots,\lfloor jn/L\rfloor}\right)\\
&\le K_0 \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{\ell=n_{(j,-)}}^{n_{(j,+)}} \left[\frac{1}{\ell-k}\frac{1}{\log\left(k-\lfloor (i-1)n/L\rfloor\right)}\frac{1}{\log\left(\lfloor jn/L\rfloor-\ell\right)}+\right.\\
&\qquad\qquad\left.+\frac{1}{(\ell-k)^{3/2}}\frac{1}{(\log(\lfloor jn/L\rfloor-\ell))^{1/p}}\right]\\
&\le K_0' \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{\ell=n_{(j,-)}}^{n_{(j,+)}} \left[\frac{1}{\ell-k}\frac{1}{\log\left(k-\lfloor (i-1)n/L\rfloor\right)}\frac{1}{\log\left(\lfloor jn/L\rfloor-\ell\right)}\right],
\end{aligned}$$

since $\frac{(\log(\lfloor jn/L\rfloor-\ell))^{1-\frac{1}{p}}\log(k-\lfloor (i-1)n/L\rfloor)}{\sqrt{\ell-k}} \le \frac{\log(n/L)^{2-1/p}}{\sqrt{2n/(\log(n))^2}} = O(1)$. Hence we have :

$$\begin{aligned}
\mathbb{E}_{\bar\nu}[\tilde M_{i,j}] &\le K_0' \sum_{k'=\lceil n/(\log(n))^2\rceil}^{\lfloor n/L\rfloor+1-\lfloor n/(\log(n))^2\rfloor} \sum_{\ell'=\lceil n/(\log(n))^2\rceil}^{\lfloor n/L\rfloor+1-\lfloor n/(\log(n))^2\rfloor} \left[\frac{1}{n/L+\ell'-k'}\frac{1}{\log(k')}\frac{1}{\log\left(\lfloor n/L\rfloor-\ell'\right)}\right]\\
&\le \frac{K_0''}{(\log(n))^2}\sum_{k',\ell'=\lceil n/(\log(n))^2\rceil}^{\lfloor n/L\rfloor-\lfloor n/(\log(n)^2)\rfloor}\frac{1}{n/L+\ell'-k'}\\
&\le \frac{K_0''n}{L(\log(n))^2}\sum_{m=2\lfloor n/(\log(n))^2\rfloor}^{2\lfloor n/L\rfloor-2\lfloor n/(\log(n)^2)\rfloor}\frac{1}{m} \le \frac{K_0''n}{L(\log(n))^2}\log(\log(n)).
\end{aligned}$$

(d) According to Markov inequality, we have :

$$\bar\nu\left(C_{i,j}\right) = O\left(\frac{1}{(\log(n))^{\frac{1}{p}-\eta}}\right).$$

(e) Let us prove that :

$$\sup_{1\le i<j<i'<j'\le L}\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}] = O\left(\frac{n^2\log(\log(n))^2}{L^2(\log(n))^4}\right). \tag{5}$$

Let $1\le i<j<i'<j'\le L$. According to proposition 3, since $\tilde M_{i,j}\circ\bar T^{-\lfloor jn/L\rfloor-\lfloor n/(\log(n))^2\rfloor}$ is constant along the unstable curves and since $\tilde M_{i',j'}\circ\bar T^{\lfloor (i'-1)n/L\rfloor+\lceil n/(\log(n))^2\rceil}$ is constant along the stable curves, we have :

$$\mathbb{E}_{\bar\nu}[\tilde M_{i,j}\tilde M_{i',j'}] \le \mathbb{E}_{\bar\nu}[\tilde M_{i,j}]\mathbb{E}_{\bar\nu}[\tilde M_{i',j'}] + \frac{n^2}{L^2}C_{(1,0)}\delta_{(1,0)}{}^{2n/(\log(n))^2}.$$

(f) Let us prove that :

$$\sup_{1\le i<i'<j'<j\le L}\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}] = O\left(\frac{n^2\log(\log(n))^2}{L^2(\log(n))^4}\right). \tag{6}$$

To prove this, we will use three times formula (1). Let $1 \leq i < i' < j' < j \leq L$. Let us denote by $\mathcal{L}$ the set of (k, k', ℓ', ℓ) such that :

$$\lfloor (i-1)n/L\rfloor + \lceil n/(\log(n))^2\rceil \leq k \leq \lfloor in/L\rfloor - \lfloor n/(\log(n))^2\rfloor \leq \lfloor (i'-1)n/L\rfloor + \lceil n/(\log(n))^2\rceil \leq k' \leq$$

$$\leq \lfloor i'n/L\rfloor - \lfloor n/(\log(n))^2\rfloor \leq \lfloor (j'-1)n/L\rfloor + \lceil n/(\log(n))^2\rceil \leq \ell' \leq \lfloor j'n/L\rfloor - \lfloor n/(\log(n))^2\rfloor \leq$$

$$\leq \lfloor (j-1)n/L\rfloor + \lceil n/(\log(n))^2\rceil \leq \ell \leq \lfloor jn/L\rfloor - \lfloor n/(\log(n))^2\rfloor$$

Applying formula (1) a first time, we get :

$$\mathbb{E}_{\bar\nu}\left[\tilde M_{i,j}\tilde M_{i',j'}\right] = \sum_{M\in[-n;n]^2}\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \bar\nu\Big(\xi_{n_{(i,-)},k-1}\cap\{S_{k'}-S_k=M\}\cap\zeta_{k'+1,n_{(i',+)}}\cap$$

$$\cap\{S_{\ell'}-S_{k'}=(0,0)\}\cap\xi_{n_{(j',-)},\ell'-1}\cap\{S_\ell-S_{\ell'}=-M\}\cap\zeta_{\ell+1,n_{(j,+)}}\Big)$$

But we have :

$$\bar\nu\Big(\xi_{n_{(i,-)},k-1}\cap\{S_{k'}-S_k=M\}\cap\zeta_{k'+1,n_{(i',+)}}\cap$$

$$\cap\{S_{\ell'}-S_{k'}=(0,0)\}\cap\xi_{n_{(j',-)},\ell'-1}\cap\{S_\ell-S_{\ell'}=-M\}\cap\zeta_{\ell+1,n_{(j,+)}}\Big)$$

$$\leq K_0 e^{-\frac{1}{2(k'-k)}a\langle M,M\rangle}\left[\frac{6K_0\bar\nu(\mathcal{A}_1)}{(k'-k)(\log(n))}+\frac{\bar\nu(\mathcal{A}_1)+\frac{6K_0}{\log(n)}(\bar\nu(\mathcal{A}_1))^{1/p}}{(k'-k)^{3/2}}\right]+\frac{K_0\bar\nu(\mathcal{A}_1)^{\frac1p}}{(k'-k)^2}$$

with $\mathcal{A}_1 := \zeta_{k'+1,n_{(i',+)}}\cap\cap\{S_{\ell'}-S_{k'}=(0,0)\}\cap\xi_{n_{(j',-)},\ell'-1}\cap\{S_\ell-S_{\ell'}=-M\}\cap\zeta_{\ell+1,n_{(j,+)}}$. Using the fact that $(\sum_{m=0}^{k-1}\varphi\circ\bar T^{-k})_{k\geq1}$ has the same distribution as $(-\sum_{m=0}^{k-1}\varphi\circ\bar T^{k})_{k\geq1}$ with respect to $\bar\nu$, we notice that :

$$\bar\nu(\mathcal{A}_1) = \bar\nu\Big(\xi_{1,n_{(j,+)}-\ell}\cap\{S_{n_{(j,+)}-\ell'+1}-S_{n_{(j,+)}-\ell+1}=M\}\cap\zeta_{n_{(j,+)}-\ell'+2,n_{(j,+)}-n_{(j,-)}-\ell'}\cap$$

$$\cap\xi_{n_{(j,+)}-n_{(i',+)}+1,n_{(j,+)}-k'}\cap\{S_{n_{(j,+)}-k'+1}-S_{n_{(j,+)}-\ell'+1}=(0,0)\}\Big).$$

Hence, applying formula (1) a second time we get :

$$\bar\nu(\mathcal{A}_1)\leq K_0 e^{-\frac{1}{2(\ell-\ell')}a\langle M,M\rangle}\left[\frac{6K_0\bar\nu(\mathcal{A}_2)}{(\ell-\ell')\log(n)}+\frac{\bar\nu(\mathcal{A}_2)+\frac{6K_0}{\log(n)}(\bar\nu(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^{3/2}}\right]+\frac{K_0\bar\nu(\mathcal{A}_2)^{\frac1p}}{(\ell-\ell')^2},$$

with $\mathcal{A}_2 := \zeta_{k'+1,n_{(i',+)}}\cap\cap\{S_{\ell'}-S_{k'}=(0,0)\}\cap\xi_{n_{(j',-)},\ell'-1}$. In particular we have :

$$\sum_{k',\ell'}\bar\nu(\mathcal{A}_2)=\mathbb{E}_{\bar\nu}[\tilde M_{i',j'}] \text{ and } \bar\nu(\mathcal{A}_2)\leq\bar\nu(\{S_{\ell'}-S_{k'}=(0,0)\})\leq\frac{6K_0}{\ell'-k'}.$$

Let us notice that, since $\bar\nu(A_1)\leq\frac{3K_0}{\ell-\ell'}\leq\frac{3K_0(\log(n))^2}{n}$, there exists some universal constant $K_3>0$ such that :

$$\bar\nu(\mathcal{A}_1)+\frac{6K_0}{\log(n)}(\bar\nu(\mathcal{A}_1))^{1/p}\leq\frac{K_3(\bar\nu(\mathcal{A}_1))^{1/p}}{\log(n)}.$$

Hence we have :

$$\mathbb{E}_{\bar\nu}\left[\tilde M_{i,j}\tilde M_{i',j'}\right] \le K_0 e^{-\frac{1}{2(k'-k)}a\langle M,M\rangle}\left[\frac{6K_0\bar\nu(\mathcal{A}_1)}{(k'-k)(\log(n))}+\frac{K_3(\bar\nu(\mathcal{A}_1))^{1/p}}{\log(n)(k'-k)^{3/2}}\right]+\frac{K_0\bar\nu(\mathcal{A}_1)^{\frac{1}{p}}}{(k'-k)^2}. \tag{7}$$

In the same way, we get :

$$\bar\nu(\mathcal{A}_1) \le K_0 e^{-\frac{1}{2(\ell-\ell')}a\langle M,M\rangle}\left[\frac{6K_0\bar\nu(\mathcal{A}_2)}{(\ell-\ell')\log(n)}+\frac{K_3(\bar\nu(\mathcal{A}_2))^{1/p}}{\log(n)(\ell-\ell')^{3/2}}\right]+\frac{K_0\bar\nu(\mathcal{A}_2)^{\frac{1}{p}}}{(\ell-\ell')^2}, \tag{8}$$

Now we enter in the most technical part of the proof. Let us notice that the following could be shortened if we were able to prove that in formulas (7) and (8), the first term is the biggest one. But, unfortunately this is not so evident. Hence, we have to estimate each term. And, actually, the estimate will depend on the term.

- First term in (7) and first term in (8).
  We have to estimate :

$$\sum_{k,k',\ell',\ell\in\mathcal{L}}\sum_{M_1,M_2} e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{a}{2}(M_1{}^2+M_2{}^2)}\frac{1}{(k'-k)(\log(n))}\frac{\bar\nu(\mathcal{A}_2)}{(\ell-\ell')\log(n)}.$$

Let us notice that :

$$\begin{aligned}
\sum_{M_1,M_2} e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{a}{2}(M_1{}^2+M_2{}^2)} &\le \left(\sum_{M_1=-\min(k'-k,\ell-\ell')}^{\min(k'-k,\ell-\ell')} e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{a}{2}(M_1{}^2)}\right)^2\\
&\le \left(1+\int_{\mathbb{R}} e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{ax^2}{2}}\,dx\right)^2\\
&\le \left(1+\sqrt{\frac{2\pi}{a}\frac{(k'-k)(\ell-\ell')}{(k'-k)+(\ell-\ell')}}\right)^2\\
&\le c\frac{2\pi}{a}\frac{(k'-k)(\ell-\ell')}{(k'-k)+(\ell-\ell')}.
\end{aligned}$$

Hence, we have to estimate :
$\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{(k'-k)(\ell-\ell')}{(k'-k)+(\ell-\ell')}\frac{1}{(k'-k)(\log(n))}\frac{\bar\nu(\mathcal{A}_2)}{(\ell-\ell')\log(n)} \le$

$$\begin{aligned}
&\le \frac{1}{(\log(n))^2}\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{\bar\nu(\mathcal{A}_2)}{(\ell-n_{(j',+)})-(k-n_{(i',-)})}\\
&\le \frac{1}{(\log(n))^2}\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{\bar\nu(\mathcal{A}_2)}{(\ell-n_{(j',+)})-(k-n_{(i',-)})}\\
&\le \frac{1}{(\log(n))^2}\left(\sum_{k',\ell'}\bar\nu(\mathcal{A}_2)\right)\sum_{k,\ell}\frac{1}{(\ell-n_{(j',-)})-(k-n_{(i,+)})}\\
&\le \frac{1}{(\log(n))^2}\mathbb{E}_{\bar\nu}[\tilde M_{i',j'}]\sum_{k,\ell}\frac{1}{(n/L+\ell-n_{(j,-)})-(k-n_{(i,-)})}\\
&\le O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right).
\end{aligned}$$

- First term in (7) and second term in (8).
  We have to estimate :

$$\sum_{k,k',\ell',\ell\in\mathcal{L}}\sum_{M_1,M_2} e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{a}{2}({M_1}^2+{M_2}^2)}\frac{1}{(k'-k)(\log(n))^2}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^{3/2}}.$$

We have :

$$\sum_{M_1,M_2} e^{-\left(\frac{1}{k'-k}+\frac{1}{\ell-\ell'}\right)\frac{a}{2}({M_1}^2+{M_2}^2)} \leq c\frac{2\pi}{a}\frac{(k'-k)(\ell-\ell')}{(k'-k)+(\ell-\ell')}.$$

Hence we have to estimate :
$\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{1}{(k'-k)+(\ell-\ell')}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}}{(\log(n))^2(\ell-\ell')^{1/2}}$

$$\begin{aligned}
&\leq \frac{1}{(\log(n))^2}\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{1}{(\ell-\ell')^{1/2}(\ell'-k')^{1/p}(n/L+\ell-n_{(j,-)})-(k-n_{(i,-)})}\\
&\leq \frac{n^2}{L^2}\frac{1}{(\log(n))^2}\frac{(\log(n))^{1+1/p}}{n^{1/2+1/p}}\sum_{k,\ell}\frac{1}{(n/L+\ell-n_{(j,-)})-(k-n_{(i,-)})}\\
&\leq \frac{n^2}{L^2}\frac{1}{(\log(n))^{1-1/p}}n^{-1/2-1/p}\frac{n}{L}\log(\log(n))\\
&\leq \frac{n^2}{L^2}\frac{1}{(\log(n))^{1-1/p}}n^{1/2-1/p}\frac{\log(\log(n))}{L}\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

since $p<2$.

- First term in (7) and third term in (8). We have to estimate :

$$\sum_{k,k',\ell',\ell\in\mathcal{L}}\sum_{M_1,M_2} e^{-\frac{1}{k'-k}\frac{a}{2}({M_1}^2+{M_2}^2)}\frac{1}{(k'-k)(\log(n))}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^2}.$$

We have :

$$\sum_{M_1,M_2} e^{-\frac{1}{k'-k}\frac{a}{2}({M_1}^2+{M_2}^2)} \leq c\frac{2(k'-k)\pi}{a}.$$

Hence we have to estimate :

$$\begin{aligned}
\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{1}{\log(n)}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^2} &\leq (6K_0)^{1/p}\sum_{k,k',\ell',\ell\in\mathcal{L}}\frac{1}{\log(n)(\ell-\ell')^2(\ell'-k')^{1/p}}\\
&\leq (6K_0)^{1/p}\frac{n^4}{L^4}\frac{1}{\log(n)}\frac{(\log(n))^{4+2/p}}{n^{2+1/p}}\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right).
\end{aligned}$$

- Second term in (7) and first term in (8).
  We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_{M}\frac{1}{\log(n)^{1+1/p}(k'-k)^{3/2}}e^{-\left(\frac{1}{k'-k}+\frac{1}{p(\ell-\ell')}\right)\frac{a}{2}\langle M,M\rangle}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^{1/p}}.$$

Since we have $\sum_M e^{-\left(\frac{1}{k'-k}+\frac{1}{p(\ell-\ell')}\right)\frac{a}{2}\langle M,M\rangle} \leq pc\frac{2\pi}{a}\frac{(\ell-\ell')(k'-k)}{(\ell-\ell')+(k'-k)}$, we have to estimate :
$\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{(\bar\nu(\mathcal{A}_2))^{1/p}(\ell-\ell')^{1-1/p}}{\log(n)^{1+1/p}(k'-k)^{1/2}}\frac{1}{(\ell-\ell')+(k'-k)}$

$$\begin{aligned}
&\leq (3K_0)^{1/p}\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{(\ell-\ell')^{1-1/p}}{(\ell'-k')^{1/p}\log(n)^{1+1/p}(k'-k)^{1/2}}\frac{1}{(\ell-\ell')+(k'-k)}\\
&\leq \frac{n^2}{L^2}\frac{n^{1-1/p}\log(n)^{1+2/p}}{\log(n)^{1+1/p}n^{1/2+1/p}}\sum_{k,\ell}\frac{1}{(n/L+\ell-n_{(j,-)})-(k-n_{(i,-)})}\\
&\leq \frac{n^2}{L^2}n^{1/2-2/p}\log(n)^{1/p}\frac{n}{L}\log(\log(n))\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

if $p<4/3$.

- Second term in (7) and second term in (8).
  We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_M\frac{1}{\log(n)^{1+1/p}(k'-k)^{3/2}}e^{-\left(\frac{1}{k'-k}+\frac{1}{p(\ell-\ell')}\right)\frac{a}{2}\langle M,M\rangle}\frac{(\bar\nu(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{3/2p}}.$$

Since we have $\sum_M e^{-\left(\frac{1}{k'-k}+\frac{1}{p(\ell-\ell')}\right)\frac{a}{2}\langle M,M\rangle} \leq pc\frac{2\pi}{a}\frac{(\ell-\ell')(k'-k)}{(\ell-\ell')+(k'-k)}$, we have to estimate :
$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{1}{\log(n)^{1+1/p}(k'-k)^{1/2}(\ell'-k')^{1/p^2}(\ell-\ell')^{3/(2p)-1}}\frac{1}{(\ell-\ell')+(k'-k)}$

$$\begin{aligned}
&\leq \frac{n^2}{L^2}\frac{1}{\log(n)^{1+1/p}}\frac{(\log(n))^{1+2/p^2+3/p-2}}{n^{1/2+1/p^2+3/(2p)-1}}\sum_{k,\ell}\frac{1}{(n/L+\ell-n_{(j,-)})-(k-n_{(i,-)})}\\
&\leq \frac{n^2}{L^2}\frac{(\log(n))^{2/p^2+2/p-2}n^{3/2-1/p^2-3/(2p)}\log(\log(n))}{L}\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

if $3/2-1/p^2-3/(2p)<0$ (this is satisfied for example if $p<\sqrt{\frac{5}{3}}$).

- Second term in (7) and third term in (8).
  We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\sum_M\frac{1}{\log(n)(k'-k)^{3/2}}e^{-\left(\frac{1}{k'-k}\right)\frac{a}{2}\langle M,M\rangle}\frac{(\bar\nu(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{2/p}}.$$

Since we have $\sum_M e^{-\frac{1}{k'-k}\frac{a}{2}\langle M,M\rangle}\leq pc\frac{2\pi}{a}(k'-k)$, we have to estimate :

$$\begin{aligned}
\sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{1}{\log(n)(k'-k)^{1/2}}\frac{(\bar\nu(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{2/p}} &\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}}\frac{1}{\log(n)(k'-k)^{1/2}(\ell'-k')^{1/p^2}(\ell-\ell')^{2/p}}\\
&\leq \frac{n^4}{L^4\log(n)}\frac{(\log(n))^{1+\frac{2}{p^2}+\frac{4}{p}}}{n^{\frac12+\frac{1}{p^2}+\frac{2}{p}}}\\
&\leq O\left(\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

if $\frac{3}{2} - \frac{1}{p^2} - \frac{2}{p} < 0$ (this is true if $p < 4/3$).

- Third term in (7) and first term in (8).
  We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \sum_M \frac{1}{\log(n)^{1/p}(k'-k)^2} e^{-\frac{1}{2p(\ell-\ell')}a\langle M,M\rangle} \frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}}{(\ell-\ell')^{1/p}}.$$

Since we have $\sum_M e^{-\frac{1}{2p(\ell-\ell')}a\langle M,M\rangle} \leq c\frac{2\pi}{a}p(\ell-\ell')$, we have to estimate :

$$\begin{aligned}
\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{(\bar{\nu}(\mathcal{A}_2))^{1/p}(\ell-\ell')^{1-1/p}}{\log(n)^{1/p}(k'-k)^2} &\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{(\ell-\ell')^{1-1/p}}{\log(n)^{1/p}(k'-k)^2(\ell'-k')^{1/p}} \\
&\leq \frac{n^4}{L^4}\frac{n^{1-1/p}}{\log(n)^{1/p}}\frac{(\log(n))^{4+2/p}}{n^{2+1/p}} \\
&\leq \frac{n^2}{L^4}n^{1-2/p}\log(n)^{4-1/p} \\
&\leq O\left(n^2\frac{(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

since $p < 2$.

- Third term in (7) and second term in (8)
  We have to estimate :

$$\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \sum_M \frac{1}{\log(n)^{1/p}(k'-k)^2} e^{-\frac{1}{2p(\ell-\ell')}a\langle M,M\rangle} \frac{(\bar{\nu}(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{3/(2p)}}.$$

Since we have $\sum_M e^{-\frac{1}{2p(\ell-\ell')}a\langle M,M\rangle} \leq c\frac{2\pi}{a}p(\ell-\ell')$, we have to estimate :
$\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{1}{\log(n)^{1/p}(k'-k)^2}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{3/(2p)-1}} \leq$

$$\begin{aligned}
&\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}} \frac{1}{\log(n)^{1/p}(k'-k)^2(\ell'-k')^{1/p^2}(\ell-\ell')^{3/(2p)-1}} \\
&\leq \frac{n^4}{L^4}\frac{1}{\log(n)^{1/p}}\frac{\log(n)^{2+2/p^2+3/p}}{n^{1+1/p^2+3/(2p)}} \\
&\leq O\left(n^2\frac{(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

since $1 - 1/p^2 - 3/(2p) < 0$ (since $p < 2$).

- Third term in (7) and third term in (8)
  We have to estimate :

$$\begin{aligned}
\sum_{(k,k',\ell',\ell)\in\mathcal{L}} \sum_M \frac{1}{(k'-k)^2}\frac{(\bar{\nu}(\mathcal{A}_2))^{1/p^2}}{(\ell-\ell')^{2/p}} &\leq \sum_{(k,k',\ell',\ell)\in\mathcal{L}} \sum_M \frac{1}{(k'-k)^2(\ell'-k')^{1/p^2}(\ell-\ell')^{2/p}} \\
&\leq \frac{n^4}{L^4}n^2\frac{(\log(n))^{4+2/p^2+4/p}}{n^{2+1/p^2+2/p}} \\
&\leq O\left(n^2\frac{(\log(\log(n)))^2}{L^2(\log(n))^4}\right),
\end{aligned}$$

if $2 - 1/p^2 - 2/p < 0$ (this is true if $p < \sqrt{\frac{3}{2}}$).

(g) In the same way, we can prove that :

$$\sup_{1\le i<i'<j<j'\le L} \mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}] = O\left(\frac{n^2\log(\log(n))^2}{L^2(\log(n))^4}\right). \tag{9}$$

(h) Using the fact that :
$\bar\nu\left(M_{i,j} > \frac{n\log(\log(n))}{L(\log(n))^{1+\eta}} \text{ and } M_{i,j} > \frac{n\log(\log(n))}{L(\log(n))^{1+\eta}}\right) \le$

$$\le \bar\nu\left(M_{i,j}M_{i',j'} > \frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^{2+2\eta}}\right) \le \frac{\mathbb{E}_{\bar\nu}[M_{i,j}M_{i',j'}]}{\frac{n^2(\log(\log(n)))^2}{L^2(\log(n))^{2+2\eta}}}$$

Hence we have :

$$\sum_{(i,j,i',j')\,:\,\#\{i,j,i',j'\}=4} \bar\nu(C_{i,j}\cap C_{i',j'}) = O\left(\frac{L^4}{(\log(n))^{2-2\eta}}\right).$$

(i) Since we have $L=\lfloor\log(\log(n))\rfloor$, we have :

$$\begin{aligned}\bar\nu\left(\left\{N_n < (1-\varepsilon)\frac{2\pi\sqrt{det(\Sigma^2)}n}{\log(n)}\right\}\right) &= O\left(\frac{L^2\log(\log(n/L))^2}{(\log(n/L))^4}\right) + O\left(\frac{L^4}{(\log(n))^{2-2\eta}}\right)\\ &= O\left(\frac{(\log(\log(n)))^4}{(\log(n))^{2-2\eta}}\right).\end{aligned}$$

With our choice, we have : $2-2\eta>1$. This completes the proof of our result of almost sure convergence.

A Proof of proposition 3

We will need some results of [?, 5]. We will use the notations of Chernov in [5]. We take k_0 large enough (as in [5]). Let us define $\mathbb{S} := \bigcup_{k\ge k_0}\{x\in\bar M \ :\ |\varphi_x| = \pi/2-\frac{1}{k^2}\}$, where φ_x is the angular measure of $(\vec n(q),\vec v)$ taken in $[-\pi/2;\pi/2]$ if $x=(q,\vec v)$. Let us consider an integer $m\ge 0$. We denote by ξ^s_m the partition of $M\setminus\left(\bigcup_{p=0}^m \bar T^{-p}(R_0\cup\mathbb{S})\right)$ in connected components. Analogously, we denote by ξ^u_m the partition of $M\setminus\left(\bigcup_{p=0}^m \bar T^{p}(R_0\cup\mathbb{S})\right)$ in connected components. Let us recall the definition of homogeneous stable curves and homogeneous unstable curves :

- A homogeneous stable curve is a C^1 curve γ of M contained in $\bar M\setminus\left(\bigcup_{m\ge0}\bar T^{-m}(R_0\cup\mathbb{S})\right)$.
- A homogeneous unstable curve is a C^1 curve γ of M contained in $\bar M\setminus\left(\bigcup_{m\ge0}\bar T^{m}(R_0\cup\mathbb{S})\right)$.

Let us consider the set Γ^s of homogeneous stable curves and the set Γ^u of homogeneous unstable curves. We recall that there exist two constants $c_1>0$ and $\delta_1\in]0;1[$ such that :

- let y and z belonging to the same homogeneous unstable curve. Then, for any integer $n\ge0$, $\bar T^{-n}(y)$ and $\bar T^{-n}(z)$ belong to a same homogeneous unstable curve and we have : $d(\bar T^{-n}(y),\bar T^{-n}(z))\le c_1{\delta_1}^n$. Moreover, for any integer $p\ge0$, y and z belong to the same atom of ξ^u_p. Moreover, if y and z belong to the same atom of ξ^s_m, then $\bar T^m(y)$ and $\bar T^m(z)$ belong to a same homogeneous unstable curve.

- let y and z belonging to the same homogeneous stable curve. Then, for any integer $n \geq 0$, $\bar{T}^n(y)$ and $\bar{T}^n(z)$ belong to a same homogeneous stable curve and we have : $d(\bar{T}^n(y), \bar{T}^n(z)) \leq c_1 {\delta_1}^n$. Moreover, for any integer $p \geq 0$, y and z belong to the same atom of ξ_p^s. Moreover, if y and z belong to the same atom of ξ_m^u, then $\bar{T}^{-m}(y)$ and $\bar{T}^{-m}(z)$ belong to a same homogeneous stable curve.

In [5], for any y, z, Chernov defines : $s_+(x,y) := \min\{n \geq 0 \ : \ y \notin \xi_n^s(x)\}$ and $s_-(x,y) := \min\{n \geq 0 \ : \ y \notin \xi_n^u(x)\}$, where $\xi_n^s(x)$ (resp. $\xi_n^u(x)$) is the atom of ξ_n^s (resp. ξ_n^u) containing the point x. Following Chernov in [5] (page 15), let us introduce the following quantities :

$$\tilde{K}_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y,z \in \gamma^u y \neq z} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y,z)}} \ \text{ and } \ \tilde{K}_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y,z \in \gamma^s y \neq z} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

We observe that we have : $\tilde{K}_f^{(i)} \leq 2\|f\|_\infty {\delta_1}^{-\eta m} + K_f^{(i)}$, with :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y,z \in \gamma^u ; y \neq z ; s_+(y,z) \geq m+1} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y,z)}} \ \text{ and } \ K_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y,z \in \gamma^s ; y \neq z ; s_-(y,z) \geq m+1} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

With these definitions, we have : $K_f^{(1)} \leq (c_1)^\eta C_f^{(\eta,u,m)}$ and $K_f^{(2)} \leq (c_1)^\eta C_f^{(\eta,s,m)}$. Let us prove the first inequality. Moreover, Chernov establishes the existence of $c_3 > 0$ and $\alpha_3 \in (0;1)$ such that, for any integer $n \geq 0$ and for any bounded $\mathbb{C}$-valued functions f and g, we have :

$$|Cov_{\bar{\nu}}(f, g \circ \bar{T}^n)| \leq c_3 \left(\|f\|_\infty \|g\|_\infty + \|f\|_\infty \tilde{K}_g^{(2)} + \|g\|_\infty \tilde{K}_f^{(1)} \right) (\alpha_3)^n$$

(cf. theorem 4.3 of [5] and the remark following this theorem).

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